Pergamon

www.elsevier.com/locate/jappmathmech

PII: S0021-8928(03)00045-5

*J. Appl. Maths Mechs*, Vol. 67, No. 2, pp. 209–213, 2003

0021-8928/03/$—see front matter

# THE STABILITY OF NON-CONSERVATIVE SYSTEMS AND AN ESTIMATE OF THE DOMAIN OF ATTRACTION†

S. A. AGAFONOV

Moscow

e-mail: agafonov_s@rambler.ru

(*Received* 23 *July* 2002)

The stability of mechanical systems, on which dissipative, gyroscopic, potential and positional non-conservative forces act, is investigated. The condition for asymptotic stability is obtained using the Lyapunov function and an estimate of the domain of attraction is also found in terms of the system being considered. A precessional system is also examined. It is shown that the condition for the asymptotic stability of a system is the condition of acceptability in the sense of the stability of a precessional system. The results obtained are applied to the problem of the stabilization, using external moments, of the steady motion of a balanced gyroscope in gimbals. © 2003 Elsevier Science Ltd. All rights reserved.

The results of this paper generalize and develop previous statements [1–8] concerning the stability of a system both by analysing the characteristic equation and constructing the Lyapunov function. We also note that a technique of structural transformations of dynamical systems, which enables one in a number of cases to eliminate terms characterizing the gyroscopic and positional non-conservative forces from the equations of motion, has been developed in [9]. The approach has been found to be effective and enables one to solve a number of problems of the stabilization of the motion of mechanical systems using a vibration.

## 1. THE STABILITY OF A SYSTEM OF GENERAL FORM

An estimate of the domain of attraction. The equations of motion of a mechanical system, on which dissipative, gyroscopic, potential and positional non-conservative forces act, can be reduced to the form

$$\ddot{x} + B\dot{x} + hG\dot{x} + Kx + Fx = X(x, \dot{x}), \quad x = (x_1, \dots, x_n)^T \tag{1.1}$$

Here, $B^T = B$, $G^T = -G$, $K^T = K$, $F^T = -F$ are constant matrices which characterize the dissipative, gyroscopic and potential forces respectively, $h > 0$ is a scalar parameter and $X(x, \dot{x})$ is a set of terms not lower than the second order in $x$ and $\dot{x}$.

The stability of the equilibrium

$$x = 0, \quad \dot{x} = 0 \tag{1.2}$$

is investigated.

We shall assume that the dissipative forces possess complete dissipation get$G \neq 0$ and that the vector function $X(x, \dot{x})$ satisfies the inequality

$$\|X(x, \dot{x})\| \le a_0(x^Tx + \dot{x}^T\dot{x}), \quad a_0 > 0, \quad \|X(x, \dot{x})\| = (X_1^2 + \dots + X_n^2)^{1/2} \tag{1.3}$$

It is also assumed that the matrix $S = G^TF + F^TG + KG - GK$ is positive definite. This condition cannot be discarded [1].

We consider the function

$$V = (\dot{x} - h^{-1}P^Tx)^T(\dot{x} - h^{-1}P^Tx) + x^T(G^TG - h^{-2}PP^T)x, \quad P = (K - F)G^{-1} + G \tag{1.4}$$

†*Prikl. Mat. Mekh.* Vol. 67, No. 2, pp. 239–243, 2003.

The inequality

$$h \rangle (c_0/g_0)^{1/2} \tag{1.5}$$

is the condition for it to be positive definite. Here, $c_0 > 0$ and $g_0 > 0$ are the largest and the smallest eigenvalue of the matrices $PP^T$ and $G^TG$.

The derivative $\dot{V}$, calculated according to (1.1) and taken with the opposite sign, reduces to the form

$$-\dot{V} = \dot{x}^T[2B + h^{-1}(P + P^T)]\dot{x} + h^{-1}x^TSx - 2h^{-1}x^TPB\dot{x} - 2(\dot{x}^T - h^{-1}x^TP)X(x, \dot{x}) \tag{1.6}$$

Using estimate (1.3), we obtain

$$|2(\dot{x}^T - h^{-1}x^TP)X(x, \dot{x})| \le 2\|\dot{x}^T - hx^TP\|\|X(x, \dot{x})\| \le a_0\gamma(h, x, \dot{x})(x^Tx + \dot{x}^T\dot{x}) \tag{1.7}$$

Here

$$\gamma(h, x, \dot{x}) = 2\|\dot{x}^T - h^{-1}x^TP\| = 2[(\dot{x} - h^{-1}P^Tx)^T(\dot{x} - h^{-1}P^Tx)]^{1/2} \tag{1.8}$$

It is obvious that

$$\gamma(h, 0, 0) = 0, \quad \gamma(h, x, \dot{x}) \ge 0$$

Henceforth, in order to simplify the form of the dependence of $\gamma$ on $h$ and $x, \dot{x}$ is not shown.

We introduce the notation

$$M = \xi E, \quad N = \eta E, \quad L = M^{1/2}\dot{x} + 1/2M^{-1/2}Q^Tx$$

$$Q = -2h^{-1}PB, \quad \xi = 2b - h^{-1}c_1 - a_0\gamma, \quad \eta = h^{-1}\mu - a_0\gamma$$

where $E$ is the unit matrix, $\mu > 0$ is the smallest eigenvalue of the matrix $S$, $b > 0$ and $b_0 > 0$ are the smallest and largest eigenvalues of the matrix $B$ respectively, and $B$, $c_1$ is the eigenvalue of the matrix $P + P^T$ with the largest modulus. Taking inequality (1.7) into account, we have

$$\begin{aligned} -\dot{V} &\ge \xi\dot{x}^T\dot{x} + \eta x^Tx + x^TQ\dot{x} = \\ &= L^TL + x^T[N - \xi^{-1}h^{-2}PB^2P^T]x \ge L^TL + (\eta - \xi^{-1}h^{-2}b_0^2c_0)x^Tx \end{aligned} \tag{1.9}$$

In (1.9), it is assumed that the matrix $M$ is positive definite, that is

$$\xi > 0 \tag{1.10}$$

The condition of negative definiteness of the function $\dot{V}$ is specified by the inequality

$$F(\gamma) \equiv a_0^2\gamma^2 - a_0(h^{-1}\mu + 2b - h^{-1}c_1)\gamma + (2b - h^{-1}c_1)h^{-1}\mu - b_0^2c_0h^{-2} > 0 \tag{1.11}$$

Since the discriminant of the polynomial $F(\gamma)$ is positive, the roots of the equation $F(\gamma) = 0$ are real. In the linear approximation ($a_0 = 0$), the positiveness of the free term in expression (1.11), that is, the inequality

$$h > h_2 = (c_1\mu + b_0^2c_0)/(2b\mu)$$

is the condition for $\dot{V}$ to be negative definite.

The coefficient of $\gamma$ in expression (1.11) cannot be positive (otherwise $h < h_1 = (c_1 - \mu)/(2b)$, which contradicts the inequality $h > h_2$, since $h_2 > h_1$). It follows from this that the roots of the polynomial $F(\gamma)$ are positive.

Putting $h_0 = \max\left[\left(\frac{c_0}{g_0}\right)^{1/2}, h_2\right]$, when $h > h_0$, we obtain $V > 0$, $\dot{V} < 0$, which implies, on the basis of Lyapunov's theorem, the asymptotic stability of system (1.1).

If $V$ is a positive-definite function and the domain $V < l$ ($l > 0$) $\dot{V} < 0$, then the domain $V < l$ lies in the domain of attraction of the equilibrium position (1.2) [10].

Taking account of expression (1.8) for $\gamma$ and, also, the form of the function $V$, which is given by expression (1.4), we note that it is possible to take $l = \gamma_1^2/4$ ($\gamma$ is the smallest positive root of the equation $F(\gamma) = 0$), since the domain $V < \gamma_1^2/4$ lies wholly in the domain $\gamma < \gamma$, where $\dot{V} < 0$.

It can be shown that inequality (1.10) follows from the inequality $\gamma < \gamma_1$. Actually, if we put $a_0\gamma_2 = 2b - h^{-1}c_1$, then the condition $\gamma < \gamma_2$ reduces to the obvious inequality

$$[(h^{-1}(\mu + c_1) - 2b)^2 + 4h^{-2}b_0^2c_0]^{1/2} > h^{-1}(\mu + c_1) - 2b$$

We will now formulate the above result in the form of a theorem.

*Theorem.* If the matrix $S = G^TF + F^TG + KG - GK$ is positive definite, then, when $h > h_0$, system (1.1) is asymptotically stable and the domain $V < \gamma_1^2/4$ lies in the domain of attraction of equilibrium position (1.2).

## 2. THE STABILITY OF A PRECESSIONAL SYSTEM

A precessional system, which is used in the applied theory of gyroscopic systems, is obtained in the first approximation from Eq. (1.1) and has the form

$$(B + hG)\dot{x} + (K + F)x = 0 \tag{2.1}$$

Consider the positive-definite function

$$V = x^T(B + hG)^T(B + hG)x \tag{2.2}$$

The derivative $\dot{V}$, calculated in accordance with system (2.1), reduces to the form

$$\dot{V} = -x^TS_1x, \quad S_1 = hS + B(K + F) + (K - F)B \tag{2.3}$$

It follows from expression (2.3) that, if the matrix $S_1$ is positive definite, the precessional system is asymptotically stable.

Since, on transferring to a precessional system, it is assumed that the parameter $h$ is fairly large, the matrix $S_1$ will be positive definite if the matrix $S$ is positive definite. Hence, if the matrix $S$ is positive definite and the parameter $h$ is fairly large, systems (1.1) and (2.1) will be simultaneously asymptotically stable.

## 3. STABILIZATION OF THE STEADY MOTION OF A GYROSCOPE IN GIMBALS

As an application of the results obtained above, we will consider the problem of the stabilization of the steady motion of a balanced gyroscope in gimbals by means of external moments. We will write the equations of motion of the system in the form of Lagrange's equations

$$\begin{gathered}\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}} - \frac{\partial\mathcal{L}}{\partial q} = M \\ q = (\alpha, \beta, \varphi), \quad \mathcal{L} = \frac{1}{2}(A_0 - C_0\sin^2\beta)\dot{\alpha}^2 + \frac{1}{2}B_0\dot{\beta}^2 + \frac{1}{2}C(\dot{\varphi} + \dot{\alpha}\sin\beta)^2 \\ A_0 = A + A_1 + A_2, \quad B_0 = A + B_1, \quad C_0 = A + A_1 - B_1\end{gathered} \tag{3.1}$$

Here, $\alpha$, $\beta$ and $\varphi$ are the angles of rotation of the external ring, the internal ring and the rotor respectively, $M = (M_\alpha, M_\beta, M_\varphi)$ is the external moment, $A_1$, $B_1$, $C_1$ are the moments of inertia of the internal ring, $A_2$ is the moment of inertia of the external ring about the axis of rotation and $A = B$ and $C$ are the equatorial and polar moments of inertia of the rotor.

On putting $M_\varphi = 0$ and eliminating the cyclic coordinate $\varphi$, the equations of motion reduce to the form

$$\begin{gathered}(A_0 - C_0\sin^2\beta)\ddot{\alpha} - C_0\dot{\alpha}\dot{\beta}\sin 2\beta + H\dot{\beta}\cos\beta = M_\alpha \\ B_0\ddot{\beta} + C_0\dot{\alpha}^2\sin\beta\cos\beta - H\dot{\alpha}\cos\beta = M_\beta\end{gathered} \tag{3.2}$$

Here, $H = C(\dot{\varphi} + \dot{\alpha}\sin\beta)$ is a cyclic constant.

Suppose the following external moments act on the system

$$M_\alpha = -d\dot{\alpha} - f\beta, \quad M_\beta = -d\dot{\beta} + f\alpha$$

where $d > 0, f > 0$ are constants. For convenience, we will introduce the dimensionless time and the parameters

$$a = C_0 A_0^{-1}, \quad c = (B_0 A_0^{-1})^{1/2}, \quad h = cHd^{-1}$$

After some reduction, system (3.2) will have the form

$$\begin{aligned} &(1 - a\sin^2\beta)a'' - a\alpha'\beta'\sin 2\beta + c\alpha' + h\beta'\cos\beta + f\beta = 0 \\ &c^2\beta'' + a\alpha'^2\sin\beta\cos\beta + c\beta' - h\alpha'\cos\beta - f\alpha = 0 \end{aligned} \tag{3.3}$$

The prime denotes a derivative with respect to $\tau$ and the earlier notation is retained for the parameter $f$. Next, we write system (3.3) in vector form

$$\begin{aligned} &x'' + Bx' + hGx' + Fx = X(x, x') \\ &x = (x_1, x_2)^T, \quad x_1 = \alpha, \quad x_2 = c\beta, \quad X = (X_1, X_2)^T \\ &B = \operatorname{diag}(c, c^{-1}), \quad G = \begin{Vmatrix} 0 & c \\ -c^{-1} & 0 \end{Vmatrix}, \quad F = \begin{Vmatrix} 0 & fc^{-1} \\ -fc^{-1} & 0 \end{Vmatrix} \\ &X_1 = \frac{a}{c}\sin 2\frac{x_2}{c}\left(1 - a\sin^2\frac{x_2}{c}\right)^{-1} x_1'x_2' - ac\sin^2\frac{x_2}{c}\left(1 - a\sin^2\frac{x_2}{c}\right)^{-1} x_1' - \\ &-\frac{h}{c}\left(1 - a\sin^2\frac{x_2}{c}\right)^{-1}\left[2\sin^2\frac{x_2}{2c} + a\sin^2\frac{x_2}{c}\left(2\cos\frac{x_2}{c} - 1\right)\right]x_2' \\ &X_2 = -\frac{a}{2c}x_2'^2\sin 2\frac{x_2}{c} - \frac{2}{c}\sin^2\frac{x_2}{2c}x_1' \end{aligned} \tag{3.4}$$

We now apply the results of Section 1 to system (3.4), which has the form of system (1.1) when $K = 0$.

The matrix $S = G^T F + F^T G = 2fc^{-2}E$ is positive definite since $f > 0$ and its eigenvalue $\mu = 2fc^{-2}$. The parameters $g_0$, $c_0$, $b$, $b_0$ and $c_1$ have the form $g_0 = c^{-2}$, $c_0 = f^2 + c^{-2}$, $b = c$, $b_0 = c^{-1}$, $c_1 = 2f$ (to be specific, it is assumed that $c < 1$).

Omitting the calculations associated with the estimation according to the norm of the non-linear terms $X_1$ and $X_2$; we present the final result

$$\|X\| = \sqrt{X_1^2 + X_2^2} \le a_0(x_1'^2 + x_2'^2 + x_2^2)$$

where

$$a_0 = \max\left\{\frac{a(1 + c^2) + c(2 - a - a^2)}{2c(1 - a)}, \ \frac{a + h(2 - a)}{2c(1 - a)}, \ \frac{2ac^2 + h(1 + a) + c(1 - a)}{4c^3(1 - a)}\right\}$$

The steady motion

$$\alpha = \beta = 0, \quad \alpha' = \beta' = 0$$

in which the planes of the rings are orthogonal is asymptotically stable when

$$h > h_0, \quad h_0 = (1 + 5f^2c^2)/(4fc^3)$$

The estimate of the domain of attraction is given by the inequality

$$\alpha'^2 + c^2\beta'^2 + c^{-2}\alpha^2 + \beta^2 + 2h^{-1}(\alpha'\beta - \alpha\beta' + f\alpha\alpha' + c^2 f\beta\beta') < \gamma_1^2/4$$

where

$$a_0\gamma_1 = fh^{-1}(c^{-2}-1)+c-[(fh^{-1}(c^{-2}-1)+c)^2-4ch^{-2}(h-h_0)]^{1/2}$$

## REFERENCES


1. AGAFONOV, S. A., The stability of the motion of non-conservative mechanical systems. *Prikl. Mat. Mekh.*, 1992, **56**, 2, 212–217.
2. KARAPETYAN, A. V., The stability of non-conservative systems. *Vestnik MGV. Ser. 1. Mathematics, Mechanics*, 1975, 4, 109–113.
3. LAKHADANOV, V. M., The effect of the structure of forces on stability of motion. *Prikl. Mat. Mekh.*, 1974, **38**, 2, 246–253.
4. MERKIN, D. P., *Gyroscopic Systems*. Nauka, Moscow, 1974.
5. FRIK, M., Zur Stabilität nichtkonservativer Lineare Systeme. *ZAMM*, 1972, **52**, 4, 47–49.
6. LI, J., On the stability of dissipative mechanical systems with circulatory forces. *ZAMM*, 1997, **48**, 1, 161–164.
7. KOSHLYAKOV, V. N., The theory of the stability of non-conservative systems. In *Navigation and Control, Inst. Matematiki Akad.* Nauk UkrSSR, Kiev 1982, 3–10.
8. GONCHARENKO, V. I., Stabilization of the motion of an unstable linear mechanical system. *Prikl. Mekh.*, 1990, **26**, 4, 79–85.
9. KOSHLYAKOV, V. N., Structural reorganizations of dynamical systems with gyroscopic forces. *Prikl. Mat. Mekh.*, 1997, **61**, 5, 774–780.
10. BARBASHIN, Ye. A., *Lyapunov Functions*. Nauka, Moscow, 1970.


*Translated by* E.L.S.